

CONTACT STRUCTURES AND GEOMETRIC TOPOLOGY

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1. INTRODUCTION

A *contact structure* on a manifold M of dimension $2n+1$ is a tangent hyperplane field, i.e. a $2n$ -dimensional sub-bundle ξ of the tangent bundle TM , satisfying the following *maximal non-integrability* condition: if ξ is written locally as the kernel of a differential 1-form α , then $\alpha \wedge (d\alpha)^n$ is required to be nowhere zero on its domain of definition. Notice that ξ determines α up to multiplication by a smooth nowhere zero function f . So the contact condition is independent of the choice of 1-form defining ξ , since $(f\alpha) \wedge d(f\alpha)^n = f^{n+1}\alpha \wedge (d\alpha)^n$. I shall always assume our contact structures to be *coorientable*, which is equivalent to saying that we can write $\xi = \ker \alpha$ with a 1-form α defined on all of M ; such an α is called a *contact form*. Then $\alpha \wedge (d\alpha)^n$ is a volume form on M , so a *contact manifold* $(M, \xi = \ker \alpha)$ has to be orientable.

The classical Darboux theorem states that any contact form α can locally be written, in suitable coordinates, as $\alpha = dz + \sum_{i=1}^n x_i dy_i$. This is one of the reasons why the most interesting aspects of contact geometry are of global nature.

Contact structures provide the mathematical language for many phenomena in classical mechanics, geometric optics and thermodynamics. Equally important for the interest in these structures are their relations with symplectic, Riemannian and complex geometry. These aspects are surveyed in [26] and [29, Chapter 1].

In the last two decades it has become increasingly apparent that contact manifolds constitute a natural framework for many problems in low-dimensional geometric topology. As hypersurfaces in symplectic 4-manifolds, 3-dimensional contact manifolds build a bridge to 4-manifold topology. This interplay between dimensions three and four has helped solve some long-standing problems in knot theory. One salient example is the result of Kronheimer–Mrowka that all non-trivial knots in the 3-sphere S^3 have the so-called property P; see Section 4.2 below. Their proof is based on a result of Eliashberg and, independently, Etnyre that any symplectic filling of a 3-dimensional contact manifold can be capped off to a closed symplectic 4-manifold.

Moreover, contact topology has inspired new approaches to some known results. Pride of place has to be given to Eliashberg’s proof [17] of Cerf’s theorem that any diffeomorphism of S^3 extends to the 4-ball, based on the classification of contact structures on S^3 ; see [29] for an exposition of that proof.

Arguably the most influential contact topological result of the last decade is due to Giroux [33], cf. [22] and [7]. He established a correspondence between contact structures on a given manifold and open book decompositions of that manifold; in

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dimension three and subject to suitable equivalences on either set of structures, this correspondence is actually one-to-one.

In the present article I want to survey a selection of these recent developments in contact topology. In Section 2 a few basic contact geometric concepts will be reviewed. I then discuss some of the results highlighted above, and others besides, from a somewhat idiosyncratic point of view. As starting point I take a surgery presentation of contact 3-manifolds due to Fan Ding and yours truly; this is the content of Section 3.

In Section 4 we then turn to applications of this structure theorem. For instance, one can use it to derive an adapted open book decomposition (see Section 4.1), thus providing an alternative proof for one direction of the Giroux correspondence in dimension three. In Section 4.2 I shall also explain in outline how symplectic caps can be constructed directly from the surgery presentation theorem, without any appeal to open books. In Section 4.3 I offer the reader an *amuse gueule* illustrating the use of contact surgery in Heegaard Floer theory. Surgery diagrams also play a supporting role in a contact topological argument for computing the diffeotopy group of the 3-manifold $S^1 \times S^2$, as will be explained in Section 4.4. An example how contact surgery can be used to detect so-called non-loose (or exceptional) Legendrian knots will be given in Section 4.5. Finally, in Section 4.6 I allow the reader a glimpse of some recent results in collaboration with Fan Ding and Otto van Koert on the diagrammatic representation of 5-dimensional contact manifolds. The Giroux correspondence reduces the description of such manifolds to that of a page of an open book (here: a 4-dimensional Stein manifold) and the monodromy of the open book. The 4-dimensional Stein manifold, in turn, can be described by a surgery picture that describes the attachment of Stein handles; the attaching circles for the 2-handles are Legendrian knots, which can be visualised in terms of their front projection from S^3 (with a point removed) to a 2-plane. In conclusion, one obtains an essentially 2-dimensional representation of a contact 5-manifold. This has implications on the classification of subcritically Stein fillable contact 5-manifolds.

2. BASIC NOTIONS AND RESULTS IN CONTACT GEOMETRY

Here I want to recall some fundamental concepts of the subject. I also mention a few classification and structure theorems necessary for understanding or putting into perspective the more recent results described in the subsequent sections.

2.1. Tight vs. overtwisted. We begin with a dichotomy of contact structures that is specific to dimension three. A smooth knot L in a contact 3-manifold (M, ξ) is called *Legendrian* if it is everywhere tangent to the contact structure. If L is homologically trivial in M , one can find an embedded surface $\Sigma \subset M$ with boundary $\partial\Sigma = L$, a so-called *Seifert surface* for L . Then L has two distinguished framings (i.e. trivialisations of its normal bundle, which can alternatively be described by a vector field along and transverse to L , or by a parallel curve obtained by pushing L in the direction of that vector field): the *surface framing*, given by a vector field tangent to the surface Σ , and the *contact framing*, given by a vector field tangent to the contact structure ξ . (The surface framing turns out to be independent of the choice of Seifert surface.)

An embedded 2-disc $\Delta \subset M$ in a contact 3-manifold (M, ξ) is called *overtwisted*, if the boundary $\partial\Delta$ is a Legendrian curve whose contact framing coincides with the surface framing. If one wishes, one may then arrange that $T_x\Delta = \xi_x$ for all $x \in \partial\Delta$.

A contact 3-manifold is called *overtwisted* if it contains an overtwisted disc; otherwise it is called *tight*. It was shown by Eliashberg [15] that the classification of overtwisted contact structures on closed 3-manifolds is a purely homotopical problem: each homotopy class of tangent 2-plane fields contains a unique overtwisted contact structure (up to isotopy). For a detailed exposition of Eliashberg's proof see [29, Chapter 4.7].

Example. Let (z, r, φ) be cylindrical coordinates on \mathbb{R}^3 . The contact structure $\xi_{\text{ot}} = \ker(\cos r \, dz + r \sin r \, d\varphi)$ is an overtwisted contact structure; each disc $\Delta_{z_0} = \{z = z_0, r \leq \pi\}$ is overtwisted.

The classification of tight contact structures, on the other hand, is a very intricate problem that has not yet been solved completely. It was shown by Bennequin [4], *avant la lettre*, that the standard contact structure $\xi_{\text{st}} = \ker(dz + x \, dy)$ on \mathbb{R}^3 is tight. We shall return to the classification of tight structures in Section 2.3.

2.2. Symplectic fillings. A contact manifold $(M^{2n-1}, \xi = \ker \alpha)$ with a *cooriented* contact structure is naturally oriented by the volume form $\alpha \wedge (d\alpha)^{n-1}$. Likewise, a symplectic manifold (W^{2n}, ω) , i.e. with ω a closed non-degenerate 2-form, is naturally oriented by the volume form ω^n .

Definition. (a) A compact symplectic manifold (W^{2n}, ω) is called a *weak (symplectic) filling* of $(M^{2n-1}, \xi = \ker \alpha)$ if $\partial W = M$ as oriented manifolds and $\omega^{n-1}|_{\xi} > 0$. Here ∂W is oriented by the ‘outward normal first’ rule.

(b) A compact symplectic manifold (W^{2n}, ω) is called a *strong (symplectic) filling* of $(M^{2n-1}, \xi = \ker \alpha)$ if $\partial W = M$ and there is a Liouville vector field Y defined near ∂W , pointing outwards along ∂W , and satisfying $\xi = \ker(i_Y \omega|_{TM})$ (as cooriented contact structure). In this case we say that (M, ξ) is the *convex* boundary of (W, ω) . Here *Liouville vector field* means that the Lie derivative $\mathcal{L}_Y \omega$ — which is the same as $d(i_Y \omega)$ because of $d\omega = 0$ and Cartan's formula $\mathcal{L}_Y = i_Y \circ d + d \circ i_Y$ — equals ω .

(c) A *Stein filling* (W, J) of (M, ξ) is a sublevel set of an exhausting strictly plurisubharmonic function on a Stein manifold such that $M = \partial W$ is the corresponding level set and ξ coincides with the complex tangencies $TM \cap J(TM)$.

For the details of (c) I refer the reader to [29, Chapter 5.4]. The following implications hold for contact structures:

$$\text{Stein fillable} \implies \text{strongly fillable} \implies \text{weakly fillable} \implies \text{tight}$$

The first implication is fairly straightforward; the second one is obvious. For the third implication and references to examples that the converse implications fail, in general, see [29, Chapter 5].

2.3. Topology of the space of contact structures. Let M be a closed (i.e. compact without boundary) odd-dimensional manifold. The space $\Xi(M)$ of contact structures on M is an open (possibly empty) subset of the space of all differential 1-forms on M . According to the Gray stability theorem [29, Theorem 2.2.2], any smooth homotopy of contact structures on M is induced by an isotopy of the manifold. So the isotopy classification of contact structures on M amounts to determining the set $\pi_0(\Xi(M))$ of path-components.

For $\dim M = 3$ it is opportune, thanks to Eliashberg's classification of over-twisted contact structures, to restrict attention to tight contact structures. Moreover, we observe that the sign of $\alpha \wedge d\alpha$ is independent of the choice of contact form α defining a given contact structure ξ . If M is oriented, we can thus speak of *positive* and *negative* contact structures. In what follows a choice of orientation for M will be understood (or specified, if M does not admit an orientation-reversing diffeomorphism), and we only consider positive contact structures.

Here are examples of 3-manifolds with a unique tight contact structure (up to isotopy); we call this structure the standard contact structure on the respective manifold and denote it by ξ_{st} :

- $S^3 \subset \mathbb{R}^4$, $\xi_{\text{st}} = \ker(x dy - y dx + z dt - t dz)$,
- $S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$, $\xi_{\text{st}} = \ker(z d\theta + x dy - y dx)$, and
- \mathbb{R}^3 , $\xi_{\text{st}} = \ker(dz + x dy)$.

Remark. The standard contact structure on S^3 , when restricted to the complement of a point, equals the standard contact structure on \mathbb{R}^3 [29, Proposition 2.1.8].

These results are due to Eliashberg, cf. [29, Chapter 4.10]. Etnyre and Honda [24] have shown that the Poincaré homology sphere P with the opposite of its natural orientation does not admit a tight contact structure. From a splitting theorem for tight contact structures due to Colin [6], cf. [10], it follows that the connected sum of two copies of P , one with its natural and one with the opposite orientation, does not admit any tight contact structure for either orientation.

On the 3-torus T^3 the contact structures $\xi_n = \ker(\sin(n\theta) dx + \cos(n\theta) dy)$, $n \in \mathbb{N}$, constitute a complete list (without repetition) of the tight contact structures *up to diffeomorphism*. The classification up to isotopy is a little more subtle, see [20]. As tangent 2-plane fields, however, the ξ_n are all homotopic to $\ker d\theta$. This is an instance of a general phenomenon for toroidal manifolds, i.e. manifolds admitting an embedding of a 2-torus that induces an injection on fundamental groups: all such manifolds admit infinitely many tight contact structures.

On the other hand, there are the following finiteness results, due to Colin–Honda–Giroux [8]:

- On each closed, oriented 3-manifold there are only finitely many homotopy classes of tangent 2-plane fields that contain a tight contact structure.
- Unless the 3-manifold is toroidal, there are only finitely many tight contact structures up to isotopy.

For $\dim M \geq 5$ there are some existence results for contact structures, cf. [27], but no complete classification on any contact manifold. An interesting result in this context is due to Seidel [52, Corollary 6.8]: the isomorphism problem for simply connected closed contact manifolds is algorithmically unsolvable — beware, though, that this does not rule out the practical solution of the problem for a given manifold.

A few things are known about the fundamental group $\pi_1(\Xi(M))$ with a chosen basepoint. For instance, for each $n \in \mathbb{N}$ the group $\pi_1(\Xi(T^3), \xi_n)$ contains an infinite cyclic subgroup [30]. Or, as shown in [12], the component of $\Xi(S^1 \times S^2)$ containing the unique tight contact structure ξ_{st} has fundamental group isomorphic to \mathbb{Z} . For results about higher homotopy groups of $\Xi(M)$ for higher-dimensional contact manifolds M see [5].

These results are intimately connected with the topology of the group $\text{Diff}_0(M)$ of diffeomorphisms of M isotopic to the identity. Write $\Xi_0(M)$ for the component

of $\Xi(M)$ containing a chosen contact structure ξ_0 . Then the map

$$\begin{array}{ccc} \mathrm{Diff}_0(M) & \longrightarrow & \Xi_0(M) \\ f & \longmapsto & Tf(\xi_0) \end{array}$$

is a Serre fibration, the homotopy lifting property being a consequence of Gray stability. The fibre $\mathrm{Cont}_0(M)$, which need not be connected, consists of those contactomorphisms (i.e. diffeomorphisms that preserve ξ_0) that are isotopic (as diffeomorphisms) to the identity. Thus, the homotopy exact sequence of this Serre fibration allows us to translate homotopical information about two of the three spaces Cont_0 , Diff_0 , and Ξ_0 into information about the third.

For the mentioned result $\pi_1(\Xi(S^1 \times S^2), \xi_{\mathrm{st}}) \cong \mathbb{Z}$, the homotopy type of the topological group $\mathrm{Diff}_0(S^1 \times S^2)$ is taken as a given. But there are also examples where contact topology can be used to extract information about the diffeomorphism group, see Section 4.4 below.

2.4. Convex hypersurfaces. The notion of a convex hypersurface has been introduced into contact geometry by Giroux [31].

Definition. A vector field X on a contact manifold (M, ξ) is called a *contact vector field* if its flow preserves the contact structure ξ . When ξ is written as $\xi = \ker \alpha$, the condition on X can be stated as $\mathcal{L}_X \alpha = \mu \alpha$ for some smooth function $\mu: M \rightarrow \mathbb{R}$.

A hypersurface $\Sigma \subset M$ is called *convex* if there is a contact vector field defined near and transverse to Σ .

Example. On $S^1 \times \mathbb{R}^2$ with contact structure $\xi = \ker(\cos \theta dx - \sin \theta dy)$, the circle $L = S^1 \times \{0\}$ is Legendrian, $X = x \partial_x + y \partial_y$ is a contact vector field, and $\Sigma = S^1 \times \partial D^2$ is a convex surface. This is actually the universal model for the neighbourhood of a Legendrian knot in a contact 3-manifold.

Convex hypersurfaces, notably in 3-dimensional contact manifolds, play an important role in the classification of contact structures and topological constructions such as surgery. The reason is the following.

Given a surface Σ in a contact 3-manifold (M, ξ) , the intersection $T\Sigma \cap \xi$ defines a singular 1-dimensional foliation Σ_ξ on Σ , the so-called *characteristic foliation*. Singularities occur at points $x \in \Sigma$ where the tangent plane $T_x \Sigma$ coincides with the contact plane ξ_x . It can be shown that the characteristic foliation Σ_ξ determines the germ of ξ near Σ . This permits, for instance, the gluing of contact manifolds along surfaces with the same characteristic foliation.

In general, the characteristic foliation is difficult to control. For convex surfaces, however, it turns out that all the essential information is contained in the *dividing set*, which is defined as the set of points in Σ where the contact vector field is contained in the contact plane; in a closed surface this set is a collection of embedded circles. The characteristic foliations of two convex surfaces with the same dividing set can be made to coincide after a C^0 -small perturbation.

2.5. Open book decompositions. Given a topological space W and a homeomorphism $\phi: W \rightarrow W$, the **mapping torus** $W(\phi)$ is the quotient space obtained from $W \times [0, 2\pi]$ by identifying $(x, 2\pi)$ with $(\phi(x), 0)$ for each $x \in W$. If W is a differential manifold and ϕ a diffeomorphism equal to the identity near the boundary ∂W , then $W(\phi)$ is in a natural way a differential manifold with boundary $\partial W \times S^1$.

According to an old theorem of Alexander, cf. [22], any closed, connected, orientable 3-manifold can be written in the form

$$M_{(\Sigma, \phi)} := \Sigma(\phi) \cup_{\text{id}} (\partial\Sigma \times D^2),$$

with Σ a compact, orientable surface with boundary; it can be arranged that the boundary $\partial\Sigma$ is connected (i.e. a single copy of S^1). Write $B \subset M$ for the link (i.e. collection of knots) corresponding to $\partial\Sigma \times \{0\}$ under this identification. Then we can define a smooth, locally trivial fibration $\mathbf{p}: M \setminus B \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$ by

$$\mathbf{p}([x, \varphi]) = [\varphi] \quad \text{for } [x, \varphi] \in \Sigma(\phi)$$

and

$$\mathbf{p}(\theta, re^{i\varphi}) = [\varphi] \quad \text{for } (\theta, re^{i\varphi}) \in \partial\Sigma \times D^2 \subset \partial\Sigma \times \mathbb{C}.$$

In other words $B \subset M$ has a tubular neighbourhood of the form $B \times D^2$, where the fibration \mathbf{p} is given by the projection onto the angular coordinate in the D^2 -factor. Such a fibration is called an *open book decomposition* with *binding* B and *pages* the closures of the fibres $\mathbf{p}^{-1}(\varphi)$. Notice that each page is a codimension 1 submanifold of M with boundary B , see Figure 1.

A submanifold $B \subset M$ that arises as the binding of an open book decomposition is called a *fibred link*.

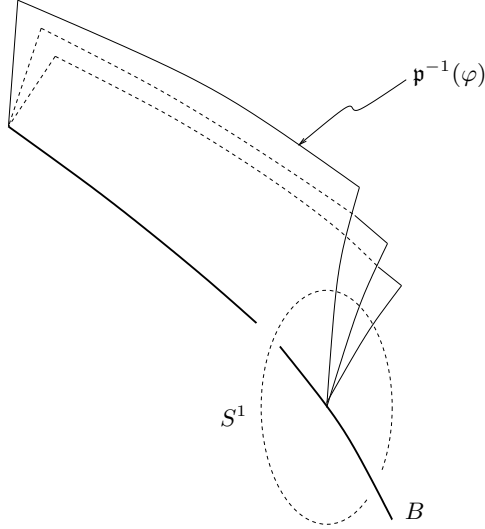


FIGURE 1. An open book near the binding.

Conversely, from an open book decomposition of M one can derive a description of M in the form $M_{(\Sigma, \phi)}$, so we may think of an open book decomposition as a pair (Σ, ϕ) . The diffeomorphism ϕ is called the *monodromy* of the open book.

In the following definition we call a contact (resp. symplectic) form on an oriented manifold *positive* if the volume form it defines on the manifold gives the positive orientation.

Definition. Let M be a manifold with an open book decomposition (B, \mathbf{p}) , where M and B are oriented. The pages of the open book are oriented consistently with

their boundary B . A contact structure $\xi = \ker \alpha$ on M defined by a positive contact form is said to be *supported* by the open book decomposition (B, \mathfrak{p}) if

- (i) the 2-form $d\alpha$ induces a positive symplectic form on each fibre of \mathfrak{p} , and
- (ii) the 1-form α induces a positive contact form on B .

Condition (i) is equivalent to the Reeb vector field R of α being positively transverse to the fibres of \mathfrak{p} . Recall that R is defined by the conditions $i_R d\alpha \equiv 0$ and $\alpha(R) \equiv 1$.

Examples. (1) The standard contact form on $S^3 \subset \mathbb{C}^2$ can be written in polar coordinates as $\alpha = r_1^2 d\varphi_1 + r_2^2 d\varphi_2$. Set $B = \{r_1 = 0\}$. Then $\mathfrak{p}: S^3 \setminus B \rightarrow S^1$, $\mathfrak{p}(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) = \varphi_1$ defines an open book whose pages are 2-discs, and whose monodromy is the identity map. This open book supports $\xi_{\text{st}} = \ker \alpha$, since α restricts to $d\varphi_2$ along the binding B , and $d\alpha$ to $r_2 dr_2 \wedge d\varphi_2$ on the tangent spaces to the pages.

(1⁺) Set $B_+ = \{r_1 r_2 = 0\}$, which is the Hopf link in S^3 . Then $\mathfrak{p}_+: S^3 \setminus B_+ \rightarrow S^1$, $\mathfrak{p}_+(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) = \varphi_1 + \varphi_2$ is an open book whose pages are annuli, and whose monodromy is a right-handed Dehn twist along the core circle of the annulus. When oriented as the boundary of a single page, the binding is a *positive* Hopf link; the annulus is called a *positive Hopf band*. For details of these claims, and the fact that this open book also supports ξ_{st} , see [29, Example 4.4.8]. Notice that the linking number of an oriented core circle of the annulus with a push-off along that annulus equals -1 .

(2) The 2-sphere S^2 admits an open book decomposition where the binding consist of the north and the south pole, the pages are half great circles between the poles, and the monodromy is the identity. When we cross this picture with S^1 we obtain an open book for $S^1 \times S^2$ with binding consisting of two circles, pages equal to annuli, and monodromy equal to the identity. The standard contact form $z d\theta + x dy - y dx$ restricts to $\pm d\theta$ along the binding, and the Reeb vector field $z \partial_\theta + x \partial_y - y \partial_x$ is transverse to the interior of the pages. So this open book supports the standard contact structure ξ_{st} .

It was shown by Thurston and Winkelnkemper [54] that any open book decomposition of a 3-manifold supports a contact structure. Giroux [33] observed that the construction carries over to higher dimensions, provided the page admits an exact symplectic form $\omega = d\beta$ which makes it a strong symplectic filling of its boundary, and the monodromy is symplectic; for details see [29, Chapter 7.3].

Giroux has also shown the converse, which is a much deeper result:

Theorem 1 (Giroux). *Every contact structure on a closed manifold is supported by an open book decomposition whose fibres are Stein manifolds, and whose monodromy is a symplectomorphism.* \square

Moreover, for 3-dimensional manifolds he has further refined this correspondence between contact structures and open books. Given an open book decomposition of a closed 3-manifold M with page Σ and monodromy ϕ , one can form a *positive stabilisation* by adding a band to Σ along $\partial\Sigma$ and composing ϕ with a right-handed Dehn twist along a simple closed curve running once over the band. This does not change the underlying 3-manifold M . Examples (1) and (1⁺) above are an instance of this phenomenon. There is then a one-to-one correspondence between contact structures on M up to isotopy and open book decompositions of M up to positive stabilisations and isotopy.

An intrinsic view of this positive stabilisation is to say that the page Σ is replaced by the plumbing of Σ with a positive Hopf band; the plumbing is done in a neighbourhood of a proper arc in Σ and in the Hopf band, respectively; see [38].

Analogously, there is a *negative stabilisation*, corresponding to a left-handed Dehn twist or a plumbing with a negative Hopf band. This will play a role in Corollary 5. The corresponding open book of S^3 has the negative Hopf link B_- as binding (which equals B_+ as a point set, but one of the two link components gets the reverse orientation), and the open book decomposition is given by $\mathbf{p}_-: S^3 \setminus B_- \rightarrow S^1$, $\mathbf{p}_-(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) = \varphi_1 - \varphi_2$.

3. A SURGERY PRESENTATION OF CONTACT 3-MANIFOLDS

3.1. Dehn surgery. Let K be a homologically trivial knot in a 3-manifold M . Write $\nu K \cong S^1 \times D^2$ for a (closed) tubular neighbourhood of K . On the boundary $\partial(\nu K) \cong T^2$ of this tubular neighbourhood there are two distinguished curves:

1. The meridian μ , defined as a simple closed curve that bounds a disc in νK .
2. The preferred longitude λ , defined as a simple closed curve parallel to K corresponding to the surface framing.

Given an orientation of M , orientations of μ and λ are chosen such that the tangent direction of μ followed by the tangent direction of λ at a transverse intersection point of μ and λ gives the orientation of T^2 (as boundary of νK).

Let p, q be coprime integers. The manifold $M_{p/q}(K)$ obtained from M by *Dehn surgery* along K with *surgery coefficient* $p/q \in \mathbb{Q} \cup \{\infty\}$ is defined as

$$M_{p/q}(K) := \overline{M \setminus \nu K} \cup_g S^1 \times D^2,$$

where the gluing map g sends the meridian $* \times \partial D^2$ to $p\mu + q\lambda$, i.e. a simple closed curve on T^2 in the class $p[\mu] + q[\lambda] \in H_1(T^2)$. The resulting manifold is determined up to diffeomorphism by the surgery coefficient (changing p, q to $-p, -q$ yields the same manifold).

For $p/q = \infty$ the surgery is trivial. If $p/q \in \mathbb{Z}$, there is a diffeomorphism $S^1 \times D^2 \rightarrow \nu K$ sending a standard longitude $\lambda_0 = S^1 \times \{*\}$ (with some point $* \in \partial D^2$) to $p\mu + q\lambda$. This implies that integer Dehn surgery can be described as cutting out $S^1 \times D^2$ and gluing in $D^2 \times S^1$ with the obvious identification of boundaries. If M is thought of as the boundary of some 4-manifold W , the surgered manifold will be the new boundary after attaching a 2-handle $D^2 \times D^2$ to W along M . For that reason, integer Dehn surgery is also called *handle surgery*.

3.2. Contact Dehn surgery. Now suppose that K is a Legendrian knot with respect to some contact structure ξ on M . Then we may replace λ by the longitude corresponding to the contact framing of K . We now consider Dehn surgery along K with coefficient p/q as before, but we define the surgery coefficient with respect to the contact framing. Notice that the two surgery coefficients differ by an integer depending only on the Legendrian knot K . This integer, the difference between the contact framing and the surface framing, is called the *Thurston–Bennequin invariant* $\text{tb}(K)$ of K . (Notice that the contact framing is defined for any Legendrian knot; the surface framing and tb are only defined for homologically trivial ones.)

It turns out that for $p \neq 0$ one can always extend the contact structure $\xi|_{M \setminus \nu K}$ to one on the surgered manifold in such a way that the extended contact structure is tight on the glued-in solid torus $S^1 \times D^2$. Moreover, subject to this tightness

condition there are but finitely many choices for such an extension, and for $p/q = 1/k$ with $k \in \mathbb{Z}$ the extension is in fact unique. These observations hinge on the fact that $\partial(\nu K)$ is a convex surface in the sense of Section 2.4. On solid tori with convex boundary condition, tight contact structures have been classified by Giroux [32] and Honda [39].

We can therefore speak sensibly of *contact $(1/k)$ -surgery*. So the contact surgeries that are well defined *and* correspond to handle surgeries are precisely the contact (± 1) -surgeries.

There is also an *ad hoc* definition for a contact 0-surgery, but here the extension over the glued-in solid torus is necessarily overtwisted, since the contact framing and the surface framing of a meridional disc coincide.

The notion of contact Dehn surgery was introduced in [9], and the following surgery presentation of contact 3-manifolds is the main result from that paper.

Theorem 2. *Let (M, ξ) be a closed, connected contact 3-manifold. Then (M, ξ) can be obtained from (S^3, ξ_{st}) by contact (± 1) -surgery along a Legendrian link.*

Sketch proof. According to a theorem of Lickorish and Wallace, M can be obtained from S^3 by surgery along some link. Since the reverse of a surgery is again a surgery, we may likewise obtain S^3 by surgery along a link in M .

It is possible to isotope that link in (M, ξ) to a Legendrian link. Then perform the surgeries as contact surgeries. This yields S^3 with some contact structure ξ' .

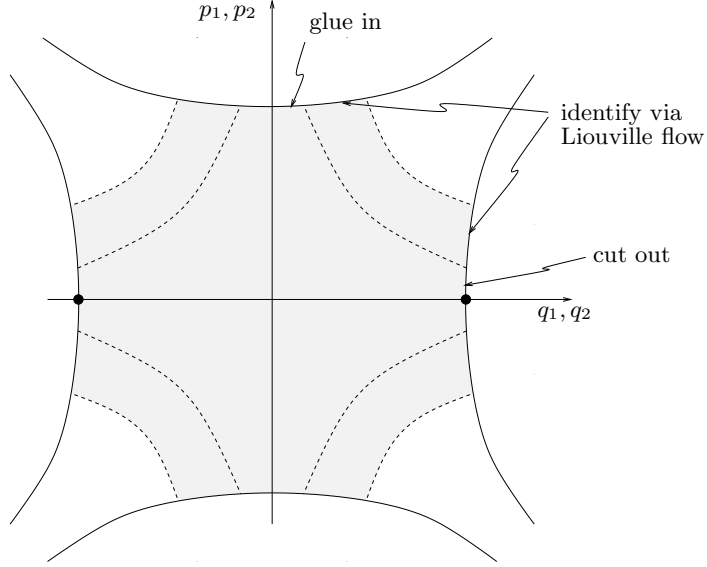
Now there is an algorithm for turning each contact surgery into a sequence of contact (± 1) -surgeries. Moreover, the contact structures on S^3 are known explicitly (the unique tight one ξ_{st} , and an overtwisted one in each homotopy class of tangent 2-plane fields). This allows one to find a further sequence of contact (± 1) -surgeries that turns (S^3, ξ') into (S^3, ξ_{st}) .

In conclusion, we can obtain (S^3, ξ_{st}) from (M, ξ) by contact (± 1) -surgery along a Legendrian link. The theorem is then a consequence of the fact that the converse of a contact (± 1) -surgery is a contact (∓ 1) -surgery. This ‘cancellation lemma’ is proved as follows, see Figure 2.

Write the Cartesian coordinates on \mathbb{R}^4 as $(\mathbf{p}, \mathbf{q}) = (p_1, p_2, q_1, q_2)$. The standard symplectic form on \mathbb{R}^4 can then be written as $\omega = d\mathbf{p} \wedge d\mathbf{q} := dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Consider the hypersurfaces $g^{-1}(\pm 1)$, where $g(\mathbf{p}, \mathbf{q}) = \mathbf{p}^2 - \mathbf{q}^2/2$, and the Liouville vector field $Y = 2\mathbf{p} \partial_{\mathbf{p}} - \mathbf{q} \partial_{\mathbf{q}}$. Notice that Y is the gradient vector field of g with respect to the standard metric on \mathbb{R}^4 . Figure 2 gives the local model for a contact (-1) -surgery along the Legendrian circle $\{\mathbf{p} = 0, \mathbf{q}^2 = 1\} \subset g^{-1}(-1)$; this follows from the neighbourhood theorem for Legendrian knots, and a computation of framings in the local model.

It is clear that the reverse surgery is the one along $\{\mathbf{p}^2 = 1, \mathbf{q} = 0\} \subset g^{-1}(1)$ in this local model, and here a computation of framings shows this to be a contact $(+1)$ -surgery. \square

Remark. In Figure 3 of Section 4.1 below we give an alternative description of contact (-1) -surgery that shows how to perform such a surgery as a symplectic handle surgery on a (weak or strong) symplectic filling, so as to obtain a filling of the surgered manifold. This type of contact surgery had been described earlier by Eliashberg [16] and Weinstein [56]. Contact $(+1)$ -surgery can be interpreted as a symplectic handle surgery on a *concave* boundary. In the ‘strong’ case this means

FIGURE 2. Contact (-1) -surgery via Liouville flow.

that we have a Liouville vector field pointing *into* the filling; in the ‘weak’ case it is a matter of changing the orientation requirements.

4. APPLICATIONS

4.1. From a surgery presentation to an open book. Given a contact 3-manifold (M, ξ) , Theorem 2 provides us with a Legendrian link $\mathbb{L} = \mathbb{L}^- \sqcup \mathbb{L}^+$ in (S^3, ξ_{st}) such that contact (± 1) -surgery along the components of \mathbb{L}^\pm yields (M, ξ) . We now want to convert this information into an open book decomposition of M supporting ξ , which can be done in two steps:

1. Find an open book for S^3 supporting ξ_{st} , such that each component of \mathbb{L} sits on a page of the open book.
2. Show that contact (-1) -surgery (resp. $(+1)$ -surgery) along a Legendrian knot sitting on a page of a supporting open book amounts to changing the monodromy by a right-handed (resp. left-handed) Dehn twist.

The first step is carried out by Plamenevskaya in [51, Proposition 4], building on work of Akbulut–Özbağcı [2]. The second step is done by Gay [25, Proposition 2.8] for contact (-1) -surgeries, and for contact surgeries of both signs by Stipsicz [53, pp. 78–79]. Their proofs rely on a result of Torisu about Heegaard splittings of contact 3-manifolds along a convex surface into two handlebodies with a tight contact structure. An alternative proof of the second step, using only local considerations, is given by Etnyre [22, Theorem 5.7]; here I give an independent and self-contained proof. Like Etnyre’s, it is done in a local model, but the surgery is described by a smooth model rather than a cut-and-paste procedure. This proof arose in discussions with Otto van Koert. Together with Niederkrüger he has extended this argument to higher dimensions; see also [1, Proposition 6.2].

In the proposition, we use the following notation: given a surface Σ and a simple closed curve $L \subset \Sigma$, we write D_L^+ for the diffeomorphism of Σ given by a right-handed Dehn twist along L ; a left-handed Dehn twist will be denoted by D_L^- .

Proposition 3. *Let $(M, \xi = \ker \alpha)$ be a contact 3-manifold with supporting open book (Σ, ϕ) , and let L be a Legendrian knot sitting on a page of this open book. Then the contact manifold obtained from (M, ξ) by contact (± 1) -surgery along L has a supporting open book $(\Sigma, \phi \circ D_L^\mp)$.*

Proof. We prove this for a contact (-1) -surgery along L ; the case of a contact $(+1)$ -surgery is completely analogous. We begin with a modified local model for a contact (-1) -surgery, see Figure 3. As in Figure 2 we consider \mathbb{R}^4 with symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q}$ and Liouville vector field $Y = 2\mathbf{p} \partial_{\mathbf{p}} - \mathbf{q} \partial_{\mathbf{q}}$. But instead of the hypersurface $g^{-1}(-1)$, we now take the hypersurface $\{\mathbf{q}^2 = 1\}$ as a model for our contact manifold in a neighbourhood of the Legendrian knot L , which we identify with $\{\mathbf{p} = 0, \mathbf{q}^2 = 1\}$. Perform the surgery along L by attaching a handle as shown in Figure 3, whose boundary is transverse to the Liouville vector field Y and hence inherits the contact form $i_Y \omega = 2\mathbf{p} d\mathbf{q} + \mathbf{q} d\mathbf{p}$.

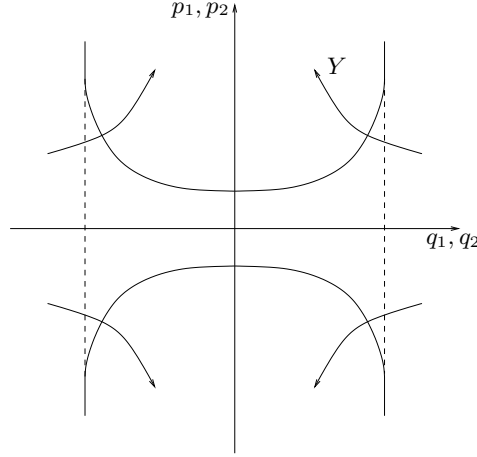


FIGURE 3. Contact (-1) -surgery via handle attachment.

Consider the map

$$\begin{aligned} \mathbf{p}: \quad \mathbb{R}^4 &\longrightarrow \mathbb{R} \\ (\mathbf{p}, \mathbf{q}) &\longmapsto \mathbf{p}\mathbf{q}. \end{aligned}$$

On the hypersurface $\{\mathbf{q}^2 = 1\}$ the Reeb vector field R of $i_Y \omega$ takes the form $R = \mathbf{q} \partial_{\mathbf{p}}$, so we have $R(\mathbf{p}) = \mathbf{q}^2 \equiv 1$ along that hypersurface, which implies that the Reeb vector field R is transverse to the fibres of the map \mathbf{p} . (These fibres, inside the hypersurface $\{\mathbf{q}^2 = 1\}$, are annuli.) Therefore, by a standard argument involving Gray stability, cf. [29, Chapter 2], we may identify a neighbourhood of $L \subset M$ with a neighbourhood $\{\mathbf{p}^2 < \varepsilon, \mathbf{q}^2 = 1\}$ in such a way that α becomes identified with $i_Y \omega$ (restricted to the tangent spaces of the hypersurface $\{\mathbf{q}^2 = 1\}$), and such that the map \mathbf{p} describes the open book $M \setminus B \rightarrow S^1$ in that neighbourhood. Notice that the Legendrian knot L lies on the page $\mathbf{p}^{-1}(0)$.

I claim that the map \mathbf{p} , restricted to the surgered hypersurface in the local model, still describes an open book supporting the contact structure after the surgery. In order to prove this claim, we need to describe the handle in the model more explicitly. Following the approach in [56], we write the surgered manifold in the model as a hypersurface $\{F(\mathbf{p}^2, \mathbf{q}^2) = 0\}$, where $F: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}$ is a smooth function with the properties

$$\begin{cases} F(0, 0) < 0, \\ \frac{\partial F}{\partial u} \geq 0, \quad \frac{\partial F}{\partial u} > 0 \text{ for } v = 0, \\ \frac{\partial F}{\partial v} \leq 0, \\ \left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2 > 0, \\ F(u, 1) = 0 \text{ for } u > \varepsilon^2/4. \end{cases}$$

With $\tilde{F}(\mathbf{p}, \mathbf{q}) := F(\mathbf{p}^2, \mathbf{q}^2)$ we have

$$d\tilde{F}(Y) = 4\mathbf{p}^2 \frac{\partial F}{\partial u} - 2\mathbf{q}^2 \frac{\partial F}{\partial v} > 0 \text{ along } \{\tilde{F} = 0\},$$

so $\{\tilde{F} = 0\}$ is indeed a hypersurface transverse to Y that coincides with $\{\mathbf{q}^2 = 1\}$ for $|\mathbf{p}| > \varepsilon/2$.

The Reeb vector field R of the contact form induced by $i_Y\omega$ on the hypersurface $\{\tilde{F} = 0\}$ is determined, up to scale, by the condition that $i_R d(i_Y\omega) = i_R\omega$ be proportional to $d\tilde{F}$. This implies that, up to a positive factor, the Reeb field is given by

$$R' := \frac{\partial F}{\partial u} \mathbf{p} \partial_{\mathbf{q}} - \frac{\partial F}{\partial v} \mathbf{q} \partial_{\mathbf{p}}.$$

From

$$R'(\mathbf{p}) = -\frac{\partial F}{\partial v} \mathbf{q}^2 + \frac{\partial F}{\partial u} \mathbf{p}^2 > 0 \text{ along } \{\tilde{F} = 0\}$$

it follows that \mathbf{p} does indeed define an open book supporting the contact structure on the surgered manifold.

It remains to verify that this surgery amounts to changing the monodromy by a right-handed Dehn twist D_L^+ . In 3-manifold topology it is well known that a Dehn twist on a splitting surface is equivalent to a surgery along the corresponding curve; this observation forms the basis for deriving a surgery presentation of a 3-manifold from a Heegaard splitting. For completeness I shall presently provide the argument. *A priori*, this only shows that the surgered manifold admits *some* open book decomposition where the monodromy has changed as described, but not that this is in fact the open book decomposition given by the map \mathbf{p} in the local model above. A result of Waldhausen [55, Lemma 3.5] comes to our rescue: any diffeomorphism of $\Sigma \times [0, 1]$ equal to the identity near the boundary is isotopic rel boundary to a fibre-preserving diffeomorphism; this implies that the monodromy is determined by a single page and the global topology. Beware that this is a result specific to dimension 3. Moreover, since I promised a self-contained proof, I show in Example (1) following this proof how to give a direct argument.

Imagine that we make an incision in our manifold M along the page Σ_0 containing L . Figure 4 shows a cross-section of this incision, orthogonal to L . In other

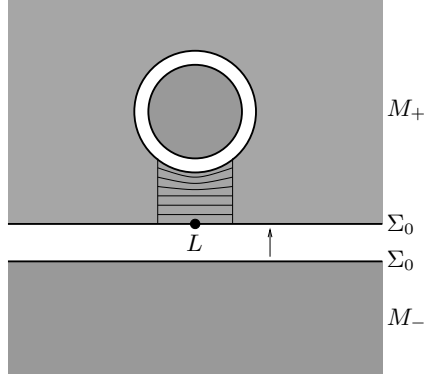


FIGURE 4. The effect of surgery on the monodromy.

words, in the figure we see L as a black dot, and the incision is seen as a horizontal cut. We think of the positive coorientation to the pages as pointing up in the figure — this is the direction of the flow transverse to the pages that determines the monodromy. With M_{\pm} we denote neighbourhoods of L on either side of Σ_0

If we want to recover the original M , we simply reglue using the identity map. In other words, in our local picture we form

$$(M_- + M_+)/(\partial M_- \ni x \sim x \in \partial M_+).$$

Now cut M_+ open along a 2-torus lying vertically over L , as shown in Figure 4. The disc seen in that figure corresponds to a solid torus T . Then the right-handed Dehn twist D_L^+ , which can be thought of as moving only points in a thin annulus around L , extends to a diffeomorphism D_+ of $M_+ \setminus T$ moving only points in the interior of the region indicated by (more or less) horizontal lines, which correspond to annuli, and acting as a right-handed Dehn twist on each of these annuli.

This diffeomorphism D_+ of $M_+ \setminus T$, and the identity map on M_- , induce a diffeomorphism from

$$(M_- + (M_+ \setminus T))/(\partial M_- \ni x \sim x \in \partial M_+)$$

to

$$(M_- + (M_+ \setminus T))/(\partial M_- \ni x \sim D_L^+(x) \in \partial M_+).$$

So we have changed the monodromy by a right-handed Dehn twist D_L^+ , at the price of cutting out a solid torus and gluing it back after we have performed the diffeomorphism D_+ on $M_+ \setminus T$. It remains to show that this cutting and regluing of T amounts to a (-1) -surgery relative to the framing of L given by the page Σ_0 (this framing coincides with the contact framing in the case of a contact structure supported by the open book).

Think of the meridian μ on ∂T as the boundary of the disc seen in Figure 4, oriented clockwise. The longitude λ corresponding to the mentioned framing is a curve parallel to L (e.g. the curve on ∂T lying vertically above L). With the standard orientation of \mathbb{R}^3 in our local model, this longitude points into the picture.

The diffeomorphism D_- has the effect of sending λ to itself and μ to $\mu + \lambda$. Thus, when we reglue the solid torus T , its meridian is glued to $\mu - \lambda$. So this is indeed a (-1) -surgery. \square

We observed in the above proof that the fibres of \mathbf{p} in the local model are annuli, so it is clear that after the contact surgery the map \mathbf{p} describes an open book whose monodromy can only have changed by a multiple of a Dehn twist along the core curve of such a fibre. The same can be said about the open book obtained from the surgery illustrated in Figure 4. Therefore, the first of the following two examples, where the monodromy directly affects the topology, implies that the change in monodromy is the same in both cases, i.e. a single Dehn twist. This argument allows us to do away with the reference to Waldhausen.

Examples. (1) Consider the open book for S^3 with binding the positive Hopf link, with pages diffeomorphic to annuli, and with monodromy a right-handed Dehn twist along the core circle of the annulus. For any $k \in \mathbb{Z}$ we now want to find the surgery necessary to turn this into an open book where the monodromy is a k -fold right-handed Dehn twist (for $k < 0$ this means a $|k|$ -fold left-handed Dehn twist), i.e. we want to add $k-1$ right-handed Dehn twists to the monodromy. According to the preceding proof, this surgery is given by regluing the solid torus T by sending its meridian to $\mu - (k-1)\lambda$. Beware, though, that λ does not give the surface framing (in S^3) of the core circle of T . By Example (1) in Section 2.5, the linking number of L with its push-off along the page is -1 . So the surface framing of the core circle of T is given by $\lambda' = \mu + \lambda$. From

$$\mu - (k-1)\lambda = k\mu - (k-1)\lambda'$$

we deduce that the required surgery is a surgery along an unknot in S^3 with surgery coefficient $-k/(k-1)$. This is the well-known surgery description of the lens space $L(k, k-1)$, cf. [37, Example 5.3.2].

For an alternative proof that the open book with page an annulus and monodromy a k -fold right-handed Dehn twist is a lens space $L(k, k-1)$ see [42]. That proof uses Brieskorn manifolds and generalises to higher dimensions.

(2) The Legendrian unknot $L = \{(e^{i\varphi}, e^{-i\varphi}) : \varphi \in [0, 2\pi]\}$ in (S^3, ξ_{st}) is the core circle in the annulus fibre $\mathbf{p}_+^{-1}(0)$ of the open book (B_+, \mathbf{p}_+) supporting ξ_{st} , see Example (1⁺) of Section 2.5. As mentioned there, the monodromy of that open book is a right-handed Dehn twist D_L^+ along L . Thus, when we perform a contact (+1)-surgery on L we obtain the contact structure supported by the open book with annulus fibres and monodromy equal to $D_L^+ \circ D_L^- = \text{id}$, which by Example (2) of Section 2.5 is the standard contact structure on $S^1 \times S^2$.

Notice that L is a standard Legendrian unknot in (S^3, ξ_{st}) with Thurston–Bennequin invariant $\text{tb}(L) = -1$ (this characterises L up to Legendrian isotopy). For alternative proofs that contact (+1)-surgery along L produces $(S^1 \times S^2, \xi_{\text{st}})$, see [13, Lemma 4.3], which uses a splitting along a convex torus, and [46, Lemma 2.5], which uses the contact invariant from Heegaard Floer theory (see Section 4.3 below). The proof in the present example is essentially equivalent to that of [53, Proposition 4.1].

The following corollary, in a slightly weaker form, was first proved by Loi–Piergallini [48]. In the form presented here, it is due to Giroux [33], cf. [22, Theorem 5.11].

Corollary 4 (Loi–Piergallini, Giroux). *A contact 3-manifold is Stein fillable if and only if it admits a supporting open book whose monodromy is a composition of right-handed Dehn twists.*

Sketch proof. Suppose the contact 3-manifold (M, ξ) is Stein fillable. According to a result of Eliashberg, cf. [36, Theorem 1.3], (M, ξ) can be obtained from a connected sum $\#S^1 \times S^2$ with its unique tight contact structure ξ_{st} by contact (-1) -surgery along a Legendrian link \mathbb{L} . There is an open book supporting ξ_{st} with trivial monodromy, just as in the preceding example. One can also construct an open book supporting ξ_{st} that contains \mathbb{L} on its pages, but may have left-handed Dehn twists in its monodromy. When we pass to a common stabilisation of these two open books, we have an open book whose monodromy can be described by right-handed Dehn twists only *and* contains \mathbb{L} on its pages. Now apply Proposition 3.

Conversely, suppose that ξ is supported by an open book (Σ, ϕ) with ϕ a composition of right-handed Dehn twists. The contact manifold described by (Σ, id) admits a Stein filling by the product $\Sigma \times D^2$; observe that

$$\partial(\Sigma \times D^2) = (\Sigma \times S^1) \cup (\partial\Sigma \times D^2) = M_{(\Sigma, \text{id})}.$$

If the Dehn twists that make up ϕ are along homologically essential curves L_i , one can realise each of them as a Legendrian curve on a page of the open book. By Proposition 3, (M, ξ) is then Stein fillable as a manifold obtained by contact (-1) -surgery on a Stein fillable manifold. If an L_i is homologically trivial in Σ , one first writes $D_{L_i}^+$ as a composition of right-handed Dehn twists along non-separating curves, and then concludes as before. \square

Remark. There is a related criterion for a contact structure to be tight. Honda–Kazez–Matić [41] introduce the notion of *right-veering* diffeomorphisms of a surface; the class of such diffeomorphisms contains those that can be written as a composition of right-handed Dehn twist. These authors show that a contact structure is tight if and only if *all* its supporting open books have right-veering monodromy.

The next topological application of contact open books, due to Giroux–Goodman [34], answers a question of Harer [38, Remark 5.1 (a)].

Corollary 5 (Giroux–Goodman). *Any fibred link in S^3 can be obtained from the unknot by finitely many plumbings and ‘deplumbings’ of Hopf bands.*

Sketch proof. Suppose $B \subset S^3$ is a fibred link, i.e. we have an open book (B, \mathbf{p}) . We formulate everything in the language of open books, where the plumbing of a Hopf band corresponds to a positive or negative stabilisation. Consider the negative stabilisation (B_-, \mathbf{p}_-) of (B, \mathbf{p}) .

In a negative Hopf band in S^3 , the two boundary circles have linking number -1 when oriented as the boundary of the band. Thus, when we orient the core circle in this band and consider a push-off of this core circle along the band, with the induced orientation, their linking number will be $+1$. It follows that in the contact structure ξ_- on S^3 supported by (B_-, \mathbf{p}_-) one can find a Legendrian unknot with $\text{tb} = +1$, which forces ξ_- to be overtwisted, since such a knot violates the Bennequin inequality [29, Theorem 4.6.36] that holds true in tight contact 3-manifolds.

Likewise, the unknot in S^3 is fibred, and after a negative stabilisation we obtain an open book (B', \mathbf{p}') supporting an overtwisted contact structure.

Once a trivialisation of the tangent bundle of S^3 has been chosen, tangent 2-plane fields on S^3 are in one-to-one correspondence with maps $S^3 \rightarrow S^2$, which are classified by the Hopf invariant, cf. [29, Chapter 4.2]. One can check that a positive stabilisation does not change the Hopf invariant of the contact structure supported by the respective open book; the examples (1) and (1^+) in Section 2.5 illustrate

this claim. A negative stabilisation, on the other hand, leads to a contact structure whose Hopf invariant is one greater.

Thus, by negatively stabilising one of (B_-, \mathfrak{p}_-) or (B', \mathfrak{p}') sufficiently often, we obtain two open books supporting overtwisted contact structures ξ_1, ξ_2 with the same Hopf invariant. So ξ_1, ξ_2 are homotopic as tangent 2-plane fields. By Eliashberg's classification of overtwisted contact structures, ξ_1 and ξ_2 are in fact isotopic as contact structures. Then the Giroux correspondence guarantees that the underlying open books become isotopic after a suitable number of further *positive* stabilisations. \square

4.2. Symplectic caps. In this section I sketch how Theorem 2 can be used to give a proof of the following theorem, due to Eliashberg [18] and Etnyre [21], and then discuss some of its topological applications. Both Eliashberg and Etnyre base their proof on an open book decomposition supporting a given contact structure; the idea for the proof indicated here belongs to Özbağcı and Stipsicz [49], see [28] for details.

Theorem 6 (Eliashberg, Etnyre). *Any weak symplectic filling (W, ω) of a contact 3-manifold (M, ξ) embeds symplectically into a closed symplectic 4-manifold.*

Sketch proof. We need to show that the given convex filling can be ‘capped off’, i.e. we need to find a concave filling of the contact 3-manifold that can be glued to the convex filling so as to produce a closed 4-manifold.

The desired cap is constructed in three stages. By Theorem 2 we know that (M, ξ) can be obtained by performing contact (± 1) -surgeries on some Legendrian link \mathbb{L} in (S^3, ξ_{st}) . For each component L_i of \mathbb{L} choose a Legendrian knot K_i in (S^3, ξ_{st}) with linking number $\text{lk}(K_i, L_i) = 1$, and $\text{lk}(K_i, L_j) = 0$ for $i \neq j$. Moreover, we require that K_i have Thurston–Bennequin invariant $\text{tb}(K_i) = 1$, which is the same as saying that contact (-1) -surgery along K_i is the same as a topological 0-surgery; such K_i can always be found. Now attach to (W, ω) the weak symplectic cobordism W_1 between (M, ξ) at the concave end and a new contact manifold (Σ^3, ξ') at the convex end, corresponding to attaching symplectic handles along the K_i . By our choices, Σ^3 will be a homology 3-sphere.

Thus, after the first step we have embedded (W, ω) symplectically into a weak filling $(W \cup_M W_1, \omega')$ of (Σ^3, ξ') . Since Σ^3 is a homology 3-sphere, the symplectic form is exact near $\Sigma^3 = \partial(W \cup_M W_1)$. This allows one to write down an explicit symplectic form on the cylinder $W_2 = \Sigma^3 \times [0, 1]$ that coincides with ω' near $\Sigma^3 \times \{0\}$ and makes $\Sigma^3 \times \{1\}$ a *strong* convex boundary (with the same induced contact structure ξ').

We now have a strong filling $(W \cup_M W_1 \cup_{\Sigma^3} (\Sigma^3 \times [0, 1]), \omega'')$ of (Σ^3, ξ') . One could then quote to a result of Gay [25] that strong fillings can be capped off; this result, however, is again based on open book decompositions. Alternatively, we appeal once more to Theorem 2, and argue as follows. Attach a (strong) symplectic cobordism corresponding to contact (-1) -surgeries that cancel the contact $(+1)$ -surgeries in a surgery presentation of (Σ^3, ξ') . The new boundary has a surgery description involving only contact (-1) -surgeries on (S^3, ξ_{st}) , which implies that it is Stein fillable. Symplectic caps for Stein fillings have been constructed by Akbulut–Özbağcı [3] and Lisca–Matić [44]. \square

This theorem has a number of topological applications, which are nicely surveyed by Etnyre [23]. For instance, Kronheimer–Mrowka [43] used it to show that every

non-trivial knot K in S^3 has the (unfortunately named) property P, which says that Dehn surgery along K with any surgery coefficient $p/q \neq \infty$ leads to a 3-manifold $S^3_{p/q}(K)$ with non-trivial fundamental group. A more palatable consequence of this fact is the Gordon–Luecke theorem, which states that knots in S^3 are determined by their complement, cf. [28]. Theorem 6 enters in the Kronheimer–Mrowka proof as follows. Given a purported counterexample, i.e. a non-trivial knot $K \subset S^3$ and some $p/q \neq \infty$ for which $\pi_1(S^3_{p/q}(K)) = \{1\}$, one constructs with the help of Theorem 6 a certain closed symplectic 4-manifold that contains, essentially, $S^3_{p/q}(K)$ as a separating hypersurface. Deep gauge theoretic results show that such a 4-manifold cannot exist.

4.3. Heegaard Floer theory. As remarked at the end of Section 3, any contact manifold obtained from a symplectically fillable contact manifold via contact (-1) -surgery will again be symplectically fillable, and hence in particular tight. It is not known, in general, whether contact (-1) -surgery on a tight contact 3-manifold will preserve tightness. For manifolds with boundary, Honda [40] has an example where tightness is destroyed by contact (-1) -surgery; for closed manifolds no such example is known.

Contact $(+1)$ -surgery may well turn a fillable contact manifold into an overtwisted one. An example is shown in Figure 5 (where the Legendrian knots in $(\mathbb{R}^3, \ker(dz + x dy)) \subset (S^3, \xi_{\text{st}})$ are represented in terms of their so-called front projection to the yz -plane; the missing x -coordinate can be recovered as the negative slope $x = -dz/dy$). The contact manifold $(S^3, \xi_{\text{st}})_{+1}(L)$ obtained from (S^3, ξ_{st}) via contact $(+1)$ -surgery on the ‘shark’ L is overtwisted. Indeed, the Legendrian knot L_{ot} bounds an overtwisted disc in $(S^3, \xi_{\text{st}})_{+1}(L)$, as is indicated on the right side of Figure 5. The Seifert surface of the Hopf link $L \sqcup L_{\text{ot}}$ shown there glues with a new meridional disc in $(S^3, \xi_{\text{st}})_{+1}(L)$ to form an embedded disc bounded by L_{ot} in the surgered manifold, and the contact framing of L_{ot} coincides with the disc framing.

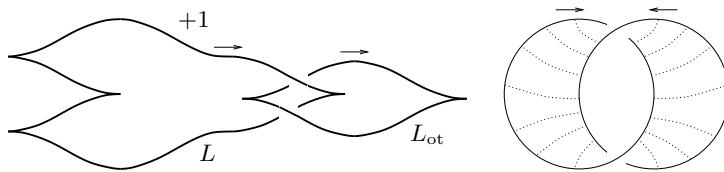


FIGURE 5. The overtwisted contact manifold $(S^3, \xi_{\text{st}})_{+1}(L)$.

On the other hand, a manifold obtained via contact $(+1)$ -surgery may also be tight, as is shown by example (2) in Section 4.1: the tight contact manifold $(S^1 \times S^2, \xi_{\text{st}})$ is obtained from (S^3, ξ_{st}) by contact $(+1)$ -surgery along a standard Legendrian unknot.

So far the most effective approach towards the question whether contact (-1) -surgery on closed contact 3-manifolds preserves tightness comes from the Heegaard Floer theory introduced by Ozsváth and Szabó [50]. Let (M, ξ) be a closed contact 3-manifold with orientation induced by the contact structure ξ . We write $-M$ for the manifold with the opposite orientation. The contact structure ξ determines a natural Spin^c structure \mathbf{t}_ξ on M . Suffice it to say here that Ozsváth and

Szabó define a contact invariant $c(M, \xi)$, which lives in the Heegaard Floer group $\widehat{HF}(-M, \mathbf{t}_\xi)$, with the following properties:

- If (M, ξ) is overtwisted, then $c(M, \xi) = 0$.
- If (M, ξ) is Stein fillable, then $c(M, \xi) \neq 0$.

If (M', ξ') is obtained from (M, ξ) by a single contact $(+1)$ -surgery (and hence (M, ξ) by contact (-1) -surgery on (M', ξ')), the cobordism W given by the contact $(+1)$ -surgery induces a homomorphism

$$F_{-W}: \widehat{HF}(-M, \mathbf{t}_\xi) \longrightarrow \widehat{HF}(-M', \mathbf{t}_{\xi'}).$$

As shown by Lisca and Stipsicz [46, Theorem 2.3], this homomorphism maps one contact invariant to the other:

$$F_{-W}(c(M, \xi)) = c(M', \xi').$$

This immediately implies the following result.

Theorem 7 (Lisca–Stipsicz). *If $c(M', \xi') \neq 0$, then $c(M, \xi) \neq 0$. In particular, (M, ξ) is tight.* \square

In a masterly series of papers, Lisca and Stipsicz have refined this approach to obtain wide-ranging existence results for tight contact structures, culminating in their paper [47], where they give a complete solution to the existence problem for tight contact structures on Seifert fibred 3-manifolds.

4.4. Diffeotopy groups. The *diffeotopy group* $\mathcal{D}(M)$ of a smooth manifold M is the quotient of the diffeomorphism group $\text{Diff}(M)$ by its normal subgroup $\text{Diff}_0(M)$ of diffeomorphisms isotopic to the identity. Alternatively, one may think of the diffeotopy group as the group $\pi_0(\text{Diff}(M))$ of path components of $\text{Diff}(M)$, since any continuous path in $\text{Diff}(M)$ can be approximated by a smooth one, i.e. an isotopy.

The theorem of Cerf (in its strong form) says that $\mathcal{D}(S^3) = \mathbb{Z}_2$, that is, up to isotopy there are only two diffeomorphisms of S^3 , the identity and an orientation reversing one. The diffeotopy groups of a number of 3-manifolds are known, for instance those of all lens spaces.

The diffeotopy group $\mathcal{D}(S^1 \times S^2)$ was shown to be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by Gluck [35]. In [12] we give a contact geometric proof of Gluck's result. The starting point for this proof is the uniqueness of the tight contact structure ξ_{st} on $S^1 \times S^2$. With Gray stability this easily translates into saying that any orientation preserving diffeomorphism of $S^1 \times S^2$ is isotopic to a contactomorphism of ξ_{st} .

In order to find an isotopy of such a contactomorphism f to one in a certain standard form, and thus to derive Gluck's theorem, one observes the effect of the contactomorphism on some Legendrian knot L in $(S^1 \times S^2, \xi_{\text{st}})$ generating the homology of $S^1 \times S^2$. This can be done in a contact surgery diagram for $(S^1 \times S^2, \xi_{\text{st}})$. The general 'Kirby moves' in such a diagram, as described in [11], then allow one to find a contact isotopy from $f(L)$ back to L . This translates into an isotopy from f to a contactomorphism fixing L . This gives one enough control over the contactomorphism to determine its isotopy type.

As an application of such methods, [12] contains examples of homologically trivial Legendrian knots in $(S^1 \times S^2, \xi_{\text{st}}) \# (S^1 \times S^2, \xi_{\text{st}})$ that cannot be distinguished by their classical invariants (i.e. the Thurston–Bennequin invariant and the rotation number, which counts the rotations of the tangent vector of the Legendrian knot

relative to a trivialisation of the contact structure over a Seifert surface) — but which may well be distinguished by performing contact (-1) -surgery on them.

4.5. Non-loose Legendrian knots. A Legendrian knot L in an overtwisted contact 3-manifold (M, ξ) is called *non-loose* or *exceptional* if the restriction of ξ to $M \setminus L$ is tight. In other words, L has to intersect each overtwisted disc Δ in (M, ξ) in such a manner that no Legendrian isotopy will allow one to separate L from Δ . This is quite a surprising phenomenon, since overtwisted discs always appear in infinite families, as in the example given in Section 2.1.

Exceptional knots were first described by Dymara [14]. For a classification of exceptional unknots in S^3 see [19, Theorem 4.7]; there is in fact a unique overtwisted contact structure on S^3 that admits exceptional unknots.

Here I want to exhibit an example, due to Lisca et al. [45, Lemma 6.1], which illustrates the use of contact surgery in detecting exceptional Legendrian knots. Figure 6 (courtesy of Paolo Lisca and András Stipsicz) shows a surgery link in (S^3, ξ_{st}) (in the front projection); the labels ± 1 indicate contact (± 1) -surgeries. The additional Legendrian knot $L(n)$, which is an unknot in S^3 , then represents a Legendrian knot in the surgered contact manifold (M, ξ) .

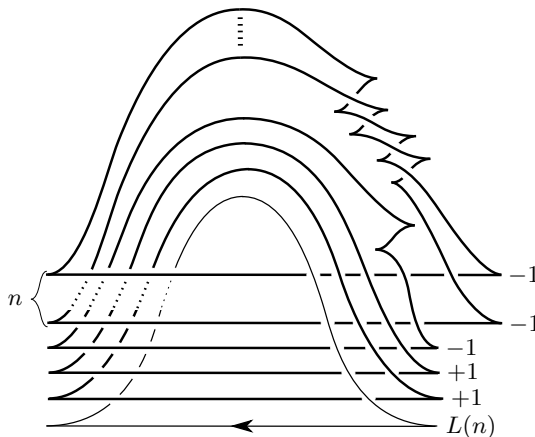


FIGURE 6. Non-loose Legendrian torus knot $T_{2,2n+1}$ in S^3 ($n \geq 1$).

By Kirby moves on this surgery diagram one can show that M is simply another copy of S^3 , and that $L(n)$ becomes the torus knot $T_{2,2n+1}$ in this 3-sphere. Moreover, with a formula given in [13, Corollary 3.6], one can easily compute the Hopf invariant of the contact structure ξ ; it turns out that this differs from the Hopf invariant ξ_{st} on S^3 . This implies that ξ and ξ_{st} are not homotopic as tangent 2-plane fields. Hence, by the uniqueness of the tight contact structure on S^3 , the contact structure ξ must be overtwisted.

We now want to convince ourselves that $L(n)$ is an exceptional knot in (S^3, ξ) . When we perform contact (-1) -surgery along $L(n)$, this cancels one of the previous $(+1)$ -surgeries. So the resulting contact manifold is the same as the one obtained from the original diagram in (S^3, ξ_{st}) , with one of the two $(+1)$ -surgery knots removed. As seen in Example (2) of Section 4.1, a single contact $(+1)$ -surgery on a Legendrian unknot as in Figure 6 results in $S^1 \times S^2$ with its unique tight (and Stein fillable) contact structure. Further contact (-1) -surgeries on this contact manifold

preserve the fillability and hence tightness of the contact structure. This implies that $T_{2,2n+1}$ is exceptional, for if there were an overtwisted disc in $S^3 \setminus T_{2,2n+1}$, it would survive to the manifold obtained by surgery along $T_{2,2n+1}$.

4.6. Diagrams for contact 5-manifolds. In the proof of Corollary 4 we alluded to a result of Eliashberg about the surgery description of Stein fillable contact 3-manifolds. That theorem is in fact a statement about the fillings; in other words, the Stein filling is obtained by attaching 1-handles to the 4-ball (resulting in a boundary connected sum of copies of $S^1 \times D^3$), and then attaching 2-handles along Legendrian knots in the boundary with framing -1 relative to the contact framing.

According to Theorem 1, any 5-dimensional contact manifold (M, ξ) is supported by an open book whose fibres are Stein surfaces. By what we just said, those fibres can be described in terms of a Kirby diagram [37] containing the information how to attach the 1- and 2-handles to the 4-ball with its standard Stein structure along its boundary (S^3, ξ_{st}) . As in Section 4.3, the pairs of attaching balls for the 1-handles and the Legendrian knots along which the 2-handles are attached can be drawn in the front projection of $(\mathbb{R}^3, \ker(dz + x dy))$ to the yz -plane.

It is not clear how to describe a general symplectic monodromy in such a diagram. Some monodromies can be encoded in the diagram, though. For instance, there are situations where one can ‘see’ Lagrangian spheres in the diagram (i.e. spheres of half the dimension of the page on whose tangent spaces the symplectic form of the page vanishes identically), and one can speak of Dehn twists along such spheres.

Here are some simple examples with trivial monodromy. Recall from the proof of Corollary 4 that the manifold M given by an open book with pages Σ and trivial monodromy is diffeomorphic to $\partial(\Sigma \times D^2)$.

Examples. (1) The diagram in Figure 7 shows a single 1-handle; this describes the 4-manifold $S^1 \times D^3$. So this is a diagram for a contact structure on $\partial(S^1 \times D^3 \times D^2) = S^1 \times S^4$.



FIGURE 7. A contact structure on $S^1 \times S^4$.

(2) The diagram in Figure 8 shows an unknot with Thurston–Bennequin invariant $\text{tb} = -1$. So this corresponds topologically to attaching a 2-handle with framing -2 relative to the surface framing (given by a spanning disc), which produces the D^2 -bundle Σ_{-2} over S^2 with Euler number -2 , see [37, Example 4.4.2]. Then $\partial(\Sigma_{-2} \times D^2)$ is the trivial S^3 -bundle over S^2 . (Observe that the S^3 -bundles over S^2 are classified by $\pi_1(\text{SO}_3) = \mathbb{Z}_2$; the non-trivial bundle is detected by the non-vanishing of the second Stiefel–Whitney class.)

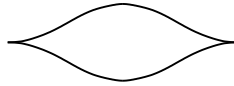


FIGURE 8. A contact structure on $S^2 \times S^3$.

(3) In Figure 9 we have an unknot with $\mathbf{tb} = -2$. So the 4-manifold encoded by this diagram is the D^2 -bundle Σ_{-3} over S^2 with Euler number -3 . It follows that $\partial(\Sigma_{-3} \times D^2)$ is the unique non-trivial S^3 -bundle over S^2 , which we write as $S^2 \tilde{\times} S^3$.

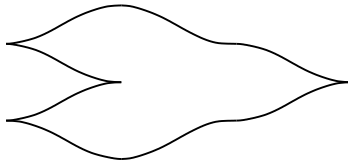


FIGURE 9. A contact structure on $S^2 \tilde{\times} S^3$.

In a forthcoming paper with Fan Ding and Otto van Koert we exploit the information contained in such diagrams, and the handle moves introduced in [12], in order to derive a number of equivalences of 5-dimensional contact manifolds. For instance, one consequence of such moves is that the contact manifold described by a single Legendrian knot L (and trivial monodromy) will always be diffeomorphic to $S^2 \times S^3$ or $S^2 \tilde{\times} S^3$, and the contact structure is completely determined by the rotation number of L .

Here is one further observation about open books with trivial monodromy. From the Seifert–van Kampen theorem one sees that the fundamental group of

$$\partial(\Sigma \times D^2) = \partial\Sigma \times D^2 \cup_{\partial} \Sigma \times S^1$$

is isomorphic to $\pi_1(\Sigma)$, since any loop in $\partial\Sigma$ is in particular a loop in Σ , and $S^1 = \partial D^2$ becomes homotopically trivial in D^2 . From a Kirby diagram for Σ one can easily read off a presentation of $\pi_1(\Sigma)$: each 1-handle gives a generator, and the attaching circles for the 2-handles provide the relations when read as words in the generators.

As observed by Cieliebak, subcritical Stein fillings (i.e. Stein fillings with no handles of maximal index) split off a D^2 -factor. Thus, contact manifolds with subcritical Stein fillings are precisely those admitting an open book with trivial monodromy.

Combining these two observations, we show in our forthcoming paper that the contactomorphism type of a subcritically fillable contact 5-manifold is, up to a certain stable equivalence, determined by its fundamental group. This result is achieved by showing how handle moves in contact surgery diagrams can be used to effect the so-called Tietze moves on the corresponding presentation of the fundamental group; any two finite presentations of a given group are related by such Tietze moves.

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